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On the Problem of Definability of the Computably Enumerable Degrees in the Difference Hierarchy

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Abstract—Questions of definability of computably enumerable degrees in the difference hierarchy (degrees of sets from finite levels of the Ershov difference hierarchy) are studied. Several approaches to the solution of this problem are outlined.

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In this paper we study questions of definability of classes of Turing degrees of n-computably enumerable (n-c. e.) sets (sets from finite levels of the Ershov difference hierarchy).

A finite level $n, n \geq 1$, of the Ershov hierarchy constitutes n-c. e. sets which can be presented in the canonical form as

$$A = \bigcup_{i=0}^{\left[\frac{n-1}{2}\right]} \{(R_{2i+1} - R_{2i})\}$$

for some c. e. sets $R_0 \subseteq R_1 \subseteq R_2 \subseteq \ldots \subseteq R_n$ (here if n is an odd number then $R_0 = \emptyset$), $\mathcal{E}_n, n \geq 1$, denotes the class of all n-c.e. sets, $\mathcal{E} = \mathcal{E}_1$ denotes the lattice of all c. e. sets.

A Turing degree $\mathbf{a}$ is n-c. e. if it contains an n-c. e. set, and it is a *properly n-c. e. degree* if it contains an n-c. e. set but no $(n-1)$-c. e. sets. The set of all n-c. e. degrees we denote by $\mathcal{D}_n$, and the set of all Turing degrees below $\mathbf{0}'$ we denote by $\mathcal{D}(\leq \mathbf{0}')$.

Definition 1. *Let $\mathcal{A}$ be a structure, and A be the universe of $\mathcal{A}$. We say that a subset B of A is definable in $\mathcal{A}$, if there is a first-order formula $\varphi(x)$ such that for any $x \in A$, $\mathcal{A} \models \varphi(x)$ if and only if $x \in B$.*

The structures of Turing degrees of n-c.e. sets are studied during last several decades and nowadays we know a lot of particular properties of these structures, but there are still several basic open problems related to their model-theoretical properties. One of them is the question of the definability of c.e. degrees in $\mathcal{D}_n$ for different $n > 1$. In this context, the issue of definability is to find formulas that produce fixed levels in the Ershov difference hierarchy. The most significant part of this problem is the question of definability of c.e. degrees in $\mathcal{D}_2$. The problem was formulated several decades ago, but nevertheless it still remains open. In this paper we present research results of recent years that has allowed to outline several possible approaches to the proof of the definability of the c.e. degrees at higher levels of the Ershov difference hierarchy.

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